Given: A swimmer is in distress 50 m from the water's edge as shown. A lifeguard is 100 m down the beach and 50 m from the water's edge.

The lifeguard can run 7 m/s across the sand and 4 m/s in shallow water. She can swim 2 m/s.

Find: The path that minimizes the time required for the lifeguard to reach the swimmer.
Solution:

If velocity is constant, then: \( \text{Time} \times \text{Velocity} = \text{Distance} \)

So \( \text{Time} = \frac{\text{Distance}}{\text{Velocity}} \)

\[
T = \frac{L_1}{V_1} + \frac{L_2}{V_2} + \frac{L_3}{V_3}
\]

Where

\[
L_1 = \sqrt{50^2 + x_1^2} \quad L_2 = \sqrt{20^2 + (x_2 - x_1)^2} \quad L_3 = \sqrt{30^2 + (100 - x_2)^2}
\]

\[
T = \frac{1}{7} \sqrt{50^2 + x_1^2} + \frac{1}{4} \sqrt{20^2 + (x_2 - x_1)^2} + \frac{1}{2} \sqrt{30^2 + (100 - x_2)^2}
\]

The objective function is time

The two design variables are \( x_1 \) and \( x_2 \)

Since there are two design variables, design space has two dimensions
From the plot, it appears that the minimum is about
\[ x_1^* = 81 \quad x_2^* = 92 \]

To find the minimum exactly, we need to find the point in design space (the plot above shows design space) where the slopes in both the \( X \) and \( Y \) directions are zero. To find this point, set both of these derivatives equal to zero and solve for \( x_1^* \) and \( y^* \).

The vector formed by these two derivatives is called a gradient:
\[ \frac{\partial T}{\partial x} = 0 \quad \leftarrow \text{Partial derivative with respect to } x \\
\frac{\partial T}{\partial y} = 0 \quad \leftarrow \text{Partial derivative with respect to } y 
\]

Here's the Mathcad solution to this system of equations:

Define objective function:

\[ T(x_1, x_2) = \frac{1}{7} \sqrt{30^2 + x_1^2} + \frac{1}{4} \sqrt{20^2 + (x_2 - x_1)^2} + \frac{1}{2} \sqrt{30^2 + (100 - x_2)^2} \]

\[ x_1 = 80 \quad x_2 = 90 \quad \text{<= Make initial guess at solution} \]

Given \( \text{<= Given statement starts Solve Block} \)

\[ \frac{d}{dx_1} T(x_1, x_2) = 0 \quad \text{<= Define first component of gradient} \]

\[ \frac{d}{dx_2} T(x_1, x_2) = 0 \quad \text{<= Define second component of gradient} \]

\[ \text{find}(x_1, x_2) = \begin{pmatrix} 81.326 \\ 92.472 \end{pmatrix} \quad \text{<= Find command calculates solution and ends Solve Block} \]
The exact solution is

\[ X_1^* = 81.336 \text{ m} \quad X_2^* = 92.472 \text{ m} \]

So, the lifeguard should run 81.34 m up the beach and wade another 11.14 m as she moves to the boundary between the shallows and deep water.

The minimum time required is \( T(X_1^*, X_2^*) = 34.83 \text{ sec} \)